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STRESS CONCENTRATION IN ULTRA-THIN COATING WITH UNDULATED SURFACE PROFILE

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Abstract. The uniaxial loading of an isotropic film-substrate system with a sinusoidal surface profile and planar interface is considered under plain strain conditions. We formulate the corresponding boundary value problem involving two-dimensional constitutive equations for bulk materials and one-dimensional equations for membrane-type surface and interface with the extra elastic constants as well as the residual surface stresses. The mixed boundary conditions consist of the generalized Young–Laplace equations and relations describing the continuous of displacements across the surface and interphase regions. Using the linear perturbation technique combined with the Goursat–Kolosov complex potentials and the superposition principle, the original boundary value problem is reduced to the analytical solution of the integral equations system.

1 INTRODUCTION

Thin films materials with a layer thickness from hundreds to a few nanometers exhibit unique physical and mechanical properties that can't be observed in bulk materials. Improved material properties are referred to significant modifications in the structure during an atomic growth process and so-called size effect related to surface stresses [1, 2, 3, 4, 5, 6]. At the stage of film deposition and subsequent thermal processing, the film surface evolves into an undulating profile [7]. Misfit stresses enhanced by a curved surface generate severe stress concentrations which may lead to a nucleation of dislocations and microcracks.

Analyzing a regular surface patterns in mono- and multilayer film coatings, we have found that even a slight undulation in surface morphology can result in a substantial increase of hoop stresses near the bottom of cavities [8, 9]. It has been shown that a stress concentration factor depends on the curvature radius and depth of cavities as well as the thickness and stiffness of film layers. However, the effect of surface elasticity on the stress state of thin film was neglected in comparison with the effect of macroscopic bulk elastic behavior. Thus, the aim of the presenting research is to extend the continuum model of coherently strained thin film deposited on a thick substrate [8] to the case of nanoscale film thickness incorporating the coupled effect of surface and interface stresses.

2 PROBLEM FORMULATION

Considering an isotropic ultra-thin film coating with a roughened surface profile and a flat interphase region under plain strain conditions, we arrive at a two-dimensional boundary value problem formulated in the terms of the complex variable $z = x_1 + ix_2$ ($i^2 = -1$ and x_1, $x_2 \in \mathbf{R}^1$ are the global Cartesian coordinates) for a strip Ω_1 of thickness h with an undulated external boundary Γ_1, bonded to a half-plain Ω_2 along a rectilinear interface Γ_2:

$$\Gamma_1 = \{z : z \equiv z_1 = x_1 + i\,[h + \varepsilon a \cos(k_a x_1)]\}\,,\ \Gamma_2 = \{z : z \equiv z_2 = x_2\}\,,\ k_a = \frac{2\pi}{a}, \tag{1}$$

$$\Omega_1 = \{z : 0 < x_2 < \varepsilon a \cos(k_a x_1)\}\,,\ \Omega_2 = \{z : x_2 < 0)\}\,. \tag{2}$$

As follows from the definition of Γ_1, the maximum deviation of the surface from the flat shape $x_2 = h$ is equal to $a\varepsilon$ where a is the wavelength of undulation and ε is a small parameter, i.e. $0 < \varepsilon \ll 1$.

According to the model of surface/interface elasticity proposed by Gurtin and Murdoch [10], the surface and interphase domains are assumed to be a negligibly thin layers adhering to the bulk phases without slipping. Here, we use the simplified constitutive equations taking into account only tangential components of the surface and the interface displacements:

$$\sigma^s_{tt}(z_j) = \gamma^0_j + (\lambda^s_j + 2\mu^s_j)\varepsilon^s_{tt}(z_j),\ z_j \in \Gamma_j,\ j = \{1, 2\}, \tag{3}$$

where ε^s_{tt} and σ^s_{tt} are the nonvanishing components of the surface strain and the Piola–Kirchhoff surface stress tensors, respectively; λ^s_j and μ^s_j are the surface Lamé constants, and γ^0_j is the residual surface stress for surface phase Γ_j.

Hooke's law for the bulk materials in the case of plane strain can be written as:

$$\begin{gathered}\sigma_{nn}(z) = (\lambda_j + 2\mu_j)\varepsilon_{nn}(z) + \lambda_j\varepsilon_{tt}(z), \\ \sigma_{tt}(z) = (\lambda_j + 2\mu_j)\varepsilon_{tt}(z) + \lambda_j\varepsilon_{nn}(z),\ \sigma_{nt}(z) = 2\mu_j\varepsilon_{nt}(z),\ z \in \Omega_j,\ j = \{1, 2\},\end{gathered} \tag{4}$$

where $\sigma_{nn}, \sigma_{tt}, \sigma_{nt}$ and $\varepsilon_{nn}, \varepsilon_{tt}, \varepsilon_{nt}$ are the components of bulk stress and strain tensors, respectively, defined in the local Cartesian coordinate system n, t; λ_j and μ_j are the Lamé constants for the bulk phase Ω_j.

The conditions of mechanical equilibrium on the curved surface Γ_1 and planar interface Γ_2 are described in terms of generalized Young–Laplace equation [1, 11]:

$$\begin{gathered}\sigma(z_1) = T^s \sigma_s(x_1), \;\; z_1 \in \Gamma_1, \\ \Delta\sigma(z_2) = \sigma^+(z_2) - \sigma^-(z_2) = i\frac{d\tau_s(x_1)}{dx_1}, \;\; z_2 \in \Gamma_2.\end{gathered} \tag{5}$$

Here and below, we use the following notations $\sigma_s(x_1) \equiv \sigma_{tt}^s(z_1)$, $\tau_s(x_1) \equiv \sigma_{tt}^s(z_2)$, $T^s(\cdot) = \kappa(x_1)(\cdot) - i\dfrac{1}{h(x_1)}\dfrac{d(\cdot)}{dx_1}$, $\sigma = \sigma_{nn} + i\sigma_{nt}$, $\sigma^\pm(z_2) = \lim\limits_{z \to z_2 \pm i0} \sigma(z)$, κ and h are the local principal curvature and the metric coefficient, accordingly.

Since we assumed that the surface phases and the bulk materials are coherent, the inseparability conditions can be defined as it follows:

$$\varepsilon_{tt}^s(z_1) = \varepsilon_{tt}(z_1), \;\; \Delta u(z_2) = u^+(z_2) - u^-(z_2) = 0, \;\; z_1 \in \Gamma_1, \;\; z_2 \in \Gamma_2, \tag{6}$$

where $u^\pm(z_2) = \lim\limits_{z \to z_2 \pm i0} u(z)$, $u = u_1 + iu_2$; u_1 and u_2 are the displacements along the corresponding coordinate axes x_1 and x_2.

At infinity, the stresses σ_{jk} $(j, k = \{1, 2\})$ in coordinates x_1, x_2 and the rotation angle ω are specified as:

$$\lim_{x_2 \to -\infty} (\sigma_{22} - i\sigma_{12}) = \lim_{x_2 \to -\infty} \omega = 0, \;\; \lim_{x_2 \to -\infty} \sigma_{11} = T_2. \tag{7}$$

A common reason for the appearance of longitudinal stress T_2 is a mismatch between the crystal lattice parameters of a film layer and a substrate.

3 BOUNDARY EQUATIONS

Following the superposition principle [7, 9, 8, 12], the solution of the boundary value problem (1)–(7), specifically the bulk stress vector $\sigma(z) = \sigma_{nn}(z) + i\sigma_{nt}(z)$ and the displacement vector $u(z) = u_1(z) + iu_2(z)$, is presented as a sum of two auxiliary problems. In the first problem, we suppose that the unknown self-balanced load p and surface stress ϑ are applied to the curvilinear boundary Γ_1 of the homogeneous half-plane $D_1^1 = \{z : x_2 < h + \varepsilon a \cos(kx_1)\}$ with the elastic properties of the film. So, the boundary condition in the terms of the stress vector σ^1 related to this problem can be written as:

$$\sigma^1(z_1) = p(z_1) + T^s\vartheta(z_1), \;\; \int_{-\infty}^{+\infty} p(\zeta)d\zeta = 0, \;\; z_1 \in \Gamma_1. \tag{8}$$

The stresses σ^1_{jk} $(j,k=\{1,2\})$ and the rotation angle ω^1 at infinity are equal to zero:

$$\lim_{x_2\to-\infty}(\sigma_{22}-i\sigma_{12})=\lim_{x_2\to-\infty}\sigma_{11}=\lim_{x_2\to-\infty}\omega=0. \tag{9}$$

The second problem describes a coupled deformation of two joint half-planes $D^2_1=\{z: x_2>0\}$ and $D^2_2=\{z: x_2<0\}$ with the elastic properties of the film and the substrate, accordingly, caused by the unknown jumps of stresses $\Delta\sigma^2$ and displacements Δu^2 along the rectilinear interface and the longitudinal stresses T_j acting in D^2_j $(j=\{1,2\})$:

$$\Delta\sigma^2(z_2)=\sigma^{2+}(z_2)-\sigma^{2-}(z_2),\ \Delta u^2(z_2)=u^{2+}(z_2)-u^{2-}(z_2),\ z_2\in\Gamma_1, \tag{10}$$

$$\lim_{x_2\to\pm\infty}\left(\sigma^2_{22}-i\sigma^2_{12}\right)=\lim_{x_2\to\pm\infty}\omega^2=0,\ \lim_{x_2\to+\infty}\sigma^2_{11}=T_1,\ \lim_{x_2\to-\infty}\sigma^2_{11}=T_2, \tag{11}$$

where $u^{2\pm}(z_2)=\lim\limits_{z\to z_2\pm i0}u^2(z)$, $\sigma^{2\pm}(z_2)=\lim\limits_{z\to z_2\pm i0}\sigma^2(z)$, $T_1=\dfrac{\mu_1(\varkappa_2+1)}{\mu_2(\varkappa_1+1)}T_2$.

The superposition principle can be expressed as:

$$G(z,\eta_j)=G^1_1(z,\eta_1)\delta_{j1}+G^2_j(z,\eta_j),\ z\in\Omega_j. \tag{12}$$

In Eq. (12), the functions $G(z,\eta_j)$, $G^1_1(z,\eta_1)$, $G^2_j(z,\eta_j)$ are equal, respectively, to $\sigma(z)$, $\sigma^1(z)$, $\sigma^2(z)$ when $\eta_j=1$ and $-2\mu_j v(z)$, $-2\mu_1 v^1(z)$, $-2\mu_j v^2(z)$ when $\eta_j=-\varkappa_j$; $\varkappa_j=3-4\nu_j$ where ν_j is Poisson's ratio of the phase Ω_j; $v(z)=du/dz$, $v^j(z)=du^j/dz$ where the derivative is taken in the direction of the axis t; δ_{j1} is the Kronecker delta and $j=\{1,2\}$.

Taking into account Eq. (12), boundary conditions (5)–(6) and constitutive equations (3)–(4) lead to the system of the boundary equations for the unknown functions p, ϑ, σ_s and τ_s:

$$\sigma^1(z_1)=p(z_1)+T^s\vartheta(z_1), \tag{13}$$

$$\Delta\sigma^2(z_2)=i\tau_s'(z_2)-\sigma^1(z_2),\ \Delta u^2(z_2)=-u^1(z_2), \tag{14}$$

$$\sigma^1(z_1)+\sigma^2(z_1)=T^s\sigma_s(z_1), \tag{15}$$

$$\vartheta(z_1)=\gamma^1_0+(\lambda^s_1+2\mu^s_1)\,\varepsilon^1_{tt}(z_1), \tag{16}$$

$$\sigma_s(z_1)=\gamma^1_0+(\lambda^s_1+2\mu^s_1)\left[\varepsilon^1_{tt}(z_1)+\varepsilon^2_{tt}(z_1)\right], \tag{17}$$

$$\tau_s(z_2)=\gamma^2_0+(\lambda^s_2+2\mu^s_2)\,\varepsilon^2_{tt}(z_2). \tag{18}$$

Thus, the solution of the general boundary value problem (1)–(7) is reduced to the solution of the system (13)–(18). To solve it, the functions σ^k and u^k $(k=\{1,2\})$ are presented in the terms of the Goursat-Kolosov complex potentials and the Muskhelishvili representation [13]. Unfortunately, it's impossible to find the exact solution of the first problem due to the curvature of the external boundary Γ_1. However, we can use the boundary perturbation method as in [1, 5, 9, 8, 12] and obtain the explicit expressions for the first-order approximation.

4 A FIRST-ORDER BOUNDARY PERTURBATION METHOD

The stress σ^k and the displacement u^k vectors are related to the Goursat-Kolosov complex potentials Φ_j^k and Υ_j^k by the following equality:

$$G_j^k(z,\eta_j) = \eta_j \Phi_j^k(w_j) + \overline{\Phi_j^k(w_j)} - \left(\Upsilon_j^k(\overline{w_j}) + \overline{\Phi_j^k(w_j)} - (w_j - \overline{w_j})\,\overline{\Phi_j^{k\prime}(w_j)}\right) e^{-2i\alpha}, \quad (19)$$

where $w_1 = z - ih$ and $w_2 = z$, α is the angle between t-axis of the local coordinates n, t and x_1-axis, the prime denotes differentiation with respect to the argument, the bar over a quantity denotes complex conjugation, Φ_1^1 and Υ_1^1 are the functions holomorphic, respectively, in D_1^1 and $\widetilde{D_1^1} = \{z : x_2 > h - \varepsilon a \cos(k_a x_1)\}$; the functions Φ_j^2 and Υ_j^2 are holomorphic in D_j^2 and D_k^2 $(j,k = \{1,2\},\ j \neq k)$. Assuming $\alpha = 0$ and $\pi/2$ in Eq. (19) and taking the conditions at infinity (9) and (11) into account, one can write:

$$\lim_{x_2\to-\infty} \Phi_1^1(z) = \lim_{x_2\to-\infty} \Upsilon_1^1(z) = 0,\ \lim_{|x_2|\to\infty} \Phi_2^j(z) = \lim_{|x_2|\to\infty} \Upsilon_2^j(z) = T_j/4. \quad (20)$$

In accordance with the first-order boundary perturbation method [5], we seek the unknown functions Φ_j^k, Υ_j^k, p and ϑ in the following form:

$$\Psi(z) = \Psi_0(z) + \varepsilon \Psi_1(z), \quad (21)$$

where Ψ could be any of the listed functions.

The boundary values of the functions Ψ_n can be presented by the linear Taylor polynomial in the vicinity of the line $x_2 = 0$, treating the real variable x_1 as a parameter:

$$\Psi_n(z_1) = \Psi_n(x_1) + i\varepsilon f(x_1)\Psi_n'(x_1),\ f(x_1) = a\cos(k_a x_1). \quad (22)$$

Also, it is possible to write the linearization in the space of the parameter ε for the subsequent functions [1, 5]:

$$e^{-2i\alpha} = 1 - 2i\varepsilon f'(x_1),\ \kappa(x_1) = \varepsilon f''(x_1),\ h^{-1}(x_1) = 1. \quad (23)$$

Substituting Eqs. (21)–(23) into Eq. (19) when $k = 1$, $z \to z_1$, $\eta_1 = 1$ and $\alpha = \alpha_1$, and equating the coeficients of ε, we obtain the first-order approximation of function $\sigma^1(z_1)$:

$$\sigma^1(z_1) = \sigma_0^1(z_1) + \varepsilon\left[\sigma_1^{1d}(z_1) + \sigma_1^{1u}(z_1)\right], \quad (24)$$

where

$$\sigma_m^1(z_1) = \Phi_{1m}^1(\xi_1^1) - \Upsilon_{1m}^1(\overline{\xi_1^1}),\ m = \{0,1\},\ \xi_1^1 = w_1(z_1),$$

$$\sigma_1^{1d}(z_1) = if(x_1)\left[\Phi_{10}^{1\prime}(x_1) + \Upsilon_{10}^{1\prime}(x_1) + \overline{\Phi_{10}^{1\prime\prime}(x_1)}\right] + 2if'(x_1)\left[\Upsilon_0(x_1) + \overline{\Phi_0(x_1)}\right]. \quad (25)$$

Introducing the piecewise function Θ_m holomorphic outside the line $\operatorname{Im} w_1 = 0$

$$\Theta_m(w_1) = \begin{cases} \Upsilon^1_{1\,m}(w_1), & \operatorname{Im} w_1 > 0, \\ \Phi^1_{1\,m}(w_1), & \operatorname{Im} w_1 < 0, \end{cases} \tag{26}$$

we reduce the boundary equation (13) to the sequence of the Riemann–Hilbert problems which solution can be written in the terms of the Cauchy-type integrals [13]:

$$\Theta_m(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{i\vartheta'_m(\zeta)}{\zeta - z}dt - \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{p_m(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{\sigma^{1d}_m(\zeta)}{\zeta - z}d\zeta. \tag{27}$$

5 SYSTEM OF INTEGRAL EQUATIONS

To solve the second problem, we pass to the limit in Eq. (19) as $z \to z_2$ when $k = 2$, $\alpha = 0$. Taking into account the boundary conditions (14) and the auxiliary functions Σ and V which are holomorphic otside the line $\operatorname{Im} z = 0$

$$\Sigma(z) = \begin{cases} \Upsilon^2_2(z) + \Phi^2_1(z), & \operatorname{Im} z > 0, \\ \Upsilon^2_1(z) + \Phi^2_2(z), & \operatorname{Im} z < 0, \end{cases} \tag{28}$$

$$V(z) = \begin{cases} \mu_1\Upsilon^2_2(z) - \mu_2\varkappa_1\Phi^2_1(z), & \operatorname{Im} z > 0, \\ \mu_2\Upsilon^2_1(z) - \mu_1\varkappa_2\Phi^2_2(z), & \operatorname{Im} z < 0, \end{cases} \tag{29}$$

we arrive to the Riemann–Hilbert problems on the boundary value of functions Σ and V. The solutions of both equations can be written as:

$$\Sigma(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{i\tau'_s(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{\sigma^1(\zeta)}{\zeta - z}d\zeta,\ V(z) = \frac{\mu_1\mu_2}{\pi i}\int\limits_{-\infty}^{+\infty} \frac{v^1(\zeta)}{\zeta - z}d\zeta. \tag{30}$$

In view of Eqs. (28) and (29), one can obtain the complex potentials Φ^2_j and Υ^2_j as it follows:

$$\begin{cases} \Phi^2_1(z) = -\Phi^2_2(z) + \Sigma(z) + T_1/4, & \operatorname{Im z} > 0, \\ \Upsilon^2_1(z) = -\Upsilon^2_2(z) + \Sigma(z) + T_1/4, & \operatorname{Im z} < 0, \end{cases} \tag{31}$$

$$\begin{cases} \Upsilon^2_2(z) = \dfrac{\mu_2\varkappa_1\Sigma(z) + V(z)}{\mu_1 + \mu_2\varkappa_1} + T_2/4, & \operatorname{Im z} > 0, \\ \Phi^2_2(z) = \dfrac{\mu_2\Sigma(z) - V(z)}{\mu_2 + \mu_1\varkappa_2} + T_2/4, & \operatorname{Im z} < 0. \end{cases} \tag{32}$$

Using the properties of the Cauchy-type integrals [13], we can rewrite the solution of the second problem in the terms of the complex potentials Φ_1^1 and Υ_1^1 if substitute Eq. (19) when $\alpha = 0$, $\eta_1 = 1$ and $\eta_1 = \kappa_1$ into Eqs. (30) for Σ and V, accordingly:

$$\Sigma(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{i\tau_s'(\zeta)}{\zeta - z} d\zeta + \begin{cases} \Upsilon_1^1(z+ih) + 2ih\overline{\Phi_1^{1\prime}(z-ih)}, & \mathrm{Im}\, z > 0, \\ \Phi_1^1(z-ih), & \mathrm{Im}\, z < 0, \end{cases} \tag{33}$$

$$V(z) = \mu_2 \begin{cases} \Upsilon_1^1(z+ih) + 2ih\overline{\Phi_1^{1\prime}(z-ih)}, & \mathrm{Im}\, z > 0 \\ -\varkappa_1 \Phi_1^1(z-ih), & \mathrm{Im}\, z < 0, \end{cases} \tag{34}$$

After that, considering Eq. (21) when $k = 2$, $z \to z_1$, $\eta_1 = 1$ and $\alpha = \alpha_1$ and taking into account Eqs. (23)–(25), we derive the first-order approximation for function $\sigma_2(z_1)$:

$$\sigma^2(z_1) = \sigma_0^2(z_1) + \varepsilon\left[\sigma_1^{2d}(z_1) + \sigma_1^{2u}(z_1)\right], \tag{35}$$

where

$$\sigma_m^2(z_1) = \Phi_{1\,m}^2(z_1) - \Upsilon_{1\,m}^2(\overline{z_1}) + 2ih\overline{\Phi_{1\,m}^{2\prime}(z_1)},\ m = \{0,1\},$$

$$\sigma_1^{2d}(z_1) = if(x_1)\left[\Phi_{1\,0}^{2\prime}(x_1) + \Upsilon_{1\,0}^{2\prime}(x_1) + 2\overline{\Phi_{1\,0}^{2\prime}(x_1)} - 2ih\overline{\Phi_{1\,0}^{2\prime\prime}(x_1)}\right] + \tag{36}$$

$$+2if'(x_1)\left[\Upsilon_{1\,0}^2(x_1) - 2ih\overline{\Phi_{1\,0}^{2\prime}(x_1)} + \overline{\Phi_{1\,0}^2(x_1)}\right].$$

Also we can derive the equalities of the first-order approximation for the strains ε_{tt}^k from (4) if we consider the approximation for the stresses σ_{tt}^k and σ_{nn}^k similar to Eq. (21):

$$\varepsilon_{tt\,m}^k(z) = \frac{1}{2(\lambda_j + \mu_j)}\left[(\lambda_j + 2\mu_j)\sigma_{tt\,m}^k - \lambda_j \sigma_{nn\,m}^k\right],\ z \in \Omega_j. \tag{37}$$

To obtain the relations of the stress tensor components $\sigma_{tt\,m}^k$ and $\sigma_{nn\,m}^k$ with the complex potentials $\Phi_{j\,m}^k$, $\Upsilon_{j\,m}^k$ of the first-order approximation, one can take the angle between t-axis and x_1-axis to be equal first α and then $\alpha + \pi/2$ in Eq. (21), and sum the results:

$$\begin{gathered} \sigma_{nn\,m}^k + i\sigma_{nt\,m} = \Phi_{j\,m}^k(z) + \overline{\Phi_{j\,m}^k(z)} - \left(\Upsilon_{j\,m}^k(\overline{z}) + \overline{\Phi_{j\,m}^k(z)} - (z - \overline{z})\,\overline{\Phi_{j\,m}^{k\prime}(z)}\right)e^{-2i\alpha}, \\ \sigma_{tt\,m}^k + \sigma_{nn\,m}^k = 4\mathrm{Re}\,\Phi_{j\,m}^k(z),\ z \in \Omega_j. \end{gathered} \tag{38}$$

Here, in Eqs. (38) and above in Eq. (37) $j, k = \{1,2\}$, $m = \{0,1\}$.

Finally, we substitute Eqs. (24), (35), (37) and (38) into Eqs. (15)–(18) and take into account Eqs. (25)–(27), (31)–(34) and (36). As a result, the system of the boundary equations (13)–(18) take the form of the integral equations system in the unknown functions p_m, ϑ'_m, $\sigma'_{s\,m}$ and $\tau'_{s\,m}$ that consists of one singular and three hypersingular equations. The hypersingular equations are obtained similar to [1, 4, 5] as the result of differentiating Eqs. (16)–(18). The kernels of the derived integral equations are the same for each step of approximation. The right-hand sides are the known continuos functions.

In the case of the zero-order approxiamtion, we arrive to the homogeneous integral equations which have only zero solution following from the physical considerations. In accordance with Eqs. (26), (27) and (31)–(34), the complex potentials of the zero-order approximation are equal:

$$\Phi^1_{1\,0}(z) = \Upsilon^1_{1\,0}(z) = 0,\ z \in \Omega_1;\ \Phi^2_{j\,0}(z) = \Upsilon^2_{1\,0}(z) = T_j/4,\ z \in \Omega_j,\ j = \{1,2\}. \qquad (39)$$

As it follows from Eqs. (12), (17)–(19) and (21), they correspond to the piecewise uniform stress state of the film coating with flat surface:

$$\sigma_{11\,0}(z) = T_j,\ z \in \Omega_2;\ \sigma_{s\,0}(z) = \gamma^0_1 + \frac{M_1(1+\varkappa_1)}{4}T_1,\ \tau_{s\,0}(z) = \gamma^0_2 + \frac{M_2(1+\varkappa_1)}{4}T_2, \quad (40)$$

where $M_1 = \dfrac{\lambda^1_s + 2\mu^1_s}{2\mu_1}$, $M_2 = \dfrac{\lambda^2_s + 2\mu^2_s}{2\mu_2}$.

We seek the solution for the first-order approximation in the following form:

$$\begin{aligned} p_1(x_1) &= A^1_{-1}e^{-ik_a x_1} + A^1_1 e^{ik_a x_1},\ \vartheta'_1(x_1) = A^2_{-1}e^{-ik_a x_1} + A^2_1 e^{ik_a x_1}, \\ \sigma'_{s\,1}(x_1) &= A^3_{-1}e^{-ik_a x_1} + A^3_1 e^{ik_a x_1},\ \tau'_{s\,1}(x_1) = A^4_{-1}e^{-ik_a x_1} + A^4_1 e^{ik_a x_1}, \end{aligned} \qquad (41)$$

Based on the properties of the Cauchy-type integrals, the system of the integral equations is reduced to the linear system of algebraic equations for the unknown complex coefficients A^j_k. After finding these coefficients, one can define the complex potentials $\Phi^k_{j\,1}$ and $\Upsilon^k_{j\,1}$ from Eqs. (26), (27), (31)–(34) and, as a consequence, the solution of the original boundary value problem (1)–(7) from Eqs. (12), (19), (21) and (41).

6 RESULTS AND CONCLUSIONS

As a numerical example, we consider the dependence of stress concentration factor $S = \max \sigma^{max}_{tt}/T_2$ on the perurbation wavelength a for $h/a = 0.15$ (Figure 1 (a)), $h/a = 0.3$ (Figure 1 (b)), surface elasticity constants $M_1 = M_2 = 0$ (solid lines), $M_1 = 0.117\,nm$, $M_2 = 0$ (dashed lines), $M_1 = M_2 = 0.117\,nm$ (dot-dashed lines), stiffness ratio $\mu_1/\mu_2 = 10$ (curves 1) and $\mu_1/\mu_2 = 0.1$ (curves 2) in case of $\varepsilon = 0.1$, $\gamma^0_1 = \gamma^0_2 = 0$, $\nu_1 = \nu_2 = 0.3$.

It is seen from the dashed lines on Figure 1 that the surface stress alone reduces the stress concentration factor. Taking into consideration the interface stress (see the dot-dashed lines) decreases the stress concentration factor as well. However, both effects decreases when the size a of the asperities increases, and the solution passes to the classical

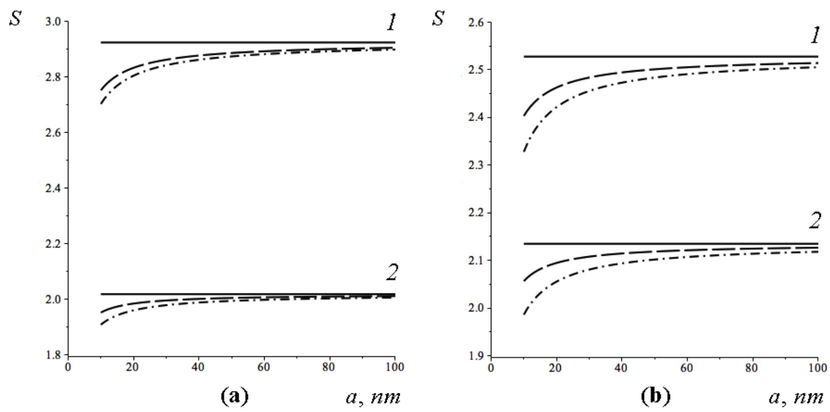

Figure 1: Stress concentration factor $S = \sigma_{tt}^{max}/T_2$ as a function of surface perturbation wavelength a.

one (i.e. to the solid lines). It should be noted that the solid lines correspond to our previous model [8]. The influence of the size of the asperities is greater for the stiffer film when $\mu_1/\mu_2 = 10$ (see the curves 1). As one can see, the stress concentration factor decreases when the stiffnes ratio μ_1/μ_2 decreases. This effect is more sensitive for the films with the smaller thickness (see Figure 1 (a)).

In this paper, we have extended our previous model of thin film with the slightly curved free surface deposited on a thick substrate to the case when the film thickness and the size of surface deffects are in nanometer range. For this purpose, we have used the surface/interface elasticity theory proposed by Gurtin and Murdoch [10], which allowed us to formulate corresponding boundary value problem involving the additional one-dimensional constitutive equations for surface phase and interphase with the extra elastic constants and the residual surface stresses. Based on the linear perturbation technique combined with the Goursat-Kolosov complex potentials and the superposition principle, the original boundary value problem has been reduced to the analytical solution of the system consisting of one singular and three hypersingular integral equations. It has been shown that the coupled effect of surface and interface stresses reduce the stress concentration factor. We have observed that this effect was more sensitive for smaller surface asperities. Finally, it should be noted that the obtained results are in a good agreement with our previous studies [1, 8].

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